Eur. Phys. J. B **58**, 461–467 (2007)
DOI: 10.1140/epjb/e2007-00237-9

THE EUROPEAN PHYSICAL JOURNAL B

Pertubation analysis of the dynamical behavior of two-junction interferometers

R. De Luca[1,a], A. Fedullo[2], and V.A. Gasanenko[3]

[1] DIIMA, Università degli Studi di Salerno, 84084 Fisciano (SA), Italy
[2] DMI, Università degli Studi di Salerno, 84084 Fisciano (SA), Italy
[3] Institute of Mathematics of National Academy of Science of Ukraine, 3 Tereshchenkivska Str. 01601 Kyiv-4, Ukraine

Received 6 February 2007 / Received in final form 4 July 2007
Published online 12 September 2007 –

Abstract. The dynamical properties of symmetric quantum interferometers with equal junctions of negligible capacitance have been studied by means of perturbation analysis in the limit of small values of the parameter β. In this limit, two characteristic time constants arise. These quantities may be linked to two different dynamical processes in the system: the first is related to the time evolution of the average superconducting phase difference across the two junctions; the second defines the time scale for flux motion. The response of the system to constant and time-dependent externally applied magnetic fields is considered and a general perturbed solution for the average superconducting phase difference and the fluxon number variable is derived to first order in β.

PACS. 74.50.+r Tunneling phenomena; point contacts, weak links, Josephson effects – 85.25.Dq Superconducting quantum interference devices (SQUIDs)

1 Introduction

The study of the dynamical properties of two junctions quantum interference models is useful in understanding the electrodynamics response of d.c. SQUIDs (Superconducting QUantum Interference Devices). The latter devices find application in various fields as ultra-high sensitive instruments [1,2] and are possible candidates for logic elements in quantum computing [3–5]. As for the latter applications, a quantum mechanical description of the SQUID system containing two overdamped Josephson junctions has only very recently been given [5]. In general, however, qubits are envisioned as superconducting loops interrupted by Josephson junctions with high capacitance and low resistance, in order to attain non dissipative Hamiltonian quantum descriptions of these systems [3,4].

The properties of two-junction interferometer models have been recently studied by means of a perturbation approach applied to the complete set of dynamical equations for the gauge-invariant superconducting phase differences across the Josephson junctions [6,7]. Following this approach, the superconducting phase variable which determines the dynamics of the whole system is the average phase φ, for which one obtains, in the limit of very small capacitance of the junctions, the following effective time evolution:

$$\frac{d\varphi}{d\tau} + (-1)^n x \sin\varphi + \pi\beta\, y^2 \sin 2\varphi = \frac{i_B}{2}, \qquad (1)$$

where n is an integer, $x = \cos\pi\psi_{ex}$, $y = \sin\pi\psi_{ex}$, ψ_{ex} being the applied magnetic flux normalized to Φ_0, the elementary flux quantum, and where $\beta = \frac{LI_J}{\Phi_0}$, L and I_J being the inductance of a single loop branch and the maximum Josephson current of both junctions, respectively, and $i_B = \frac{I_B}{I_J}$, with I_B the bias current. In equation (1) $\tau = \frac{2\pi RI_J}{\Phi_0}t$, so that $\tau_\varphi = \frac{\Phi_0}{2\pi RI_J}$ is the characteristic time constant for the average phase φ, where R is the resistive parameter of both junctions. To this dynamical equation one adds the perturbed solution, to first order in β, of the normalized flux variable ψ:

$$\psi = \psi_{ex} - 2\,(-1)^n\,\beta\, y \cos\varphi, \qquad (2)$$

where one implicitly assumes that the characteristic time for flux motion τ_ψ in the system is much lower then τ_φ, which then remains the only relevant time constant for the system. Nevertheless, in order to account for the interplay of both physical processes described by the complete model of a two-junction quantum interferometer, namely, flux motion, or dynamics of the flux number ψ, and time evolution of the average phase φ, it is necessary to consider the values of both time constants, τ_ψ and τ_φ, which in general do not necessarily differ by orders of magnitude.

In the present work we thus develop a perturbation analysis in which the time constants for flux motion and for evolution of the average superconducting phase are initially considered to be comparable, in such a way that the interplay between the two dynamical processes can be

[a] e-mail: rdeluca@unisa.it

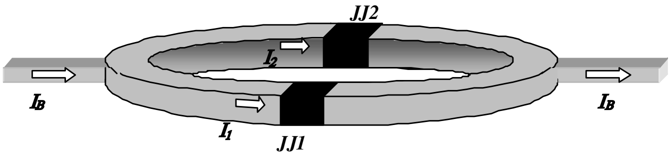

Fig. 1. A schematic representation of a symmetric quantum interferometer with two equal junctions of negligible capacitance.

studied along with the transient solution of the system's dynamics. Only after having specified the role of these two time constants, we analyze the system in the usual limit ($\tau_\psi \ll \tau_\varphi$) both in the presence of a constant and of an oscillating applied field. Therefore, in the next section we start from the complete model for a symmetric quantum interferometer with two equal junctions of negligible capacitance and indicate how the perturbation analysis can be carried out to account both regular and singular parts of the solution. In the third section the case of a constant applied flux ψ_{ex} is considered and the perturbed solution for ψ and the effective time evolution equation for φ are found to first order in the parameter β. The analysis performed for constant ψ_{ex} is extended to the case the externally applied magnetic flux varies sinusoidally with respect to time in the fourth section. Conclusions are drawn in the last section.

2 Perturbation analysis

Let us consider a symmetric two junction interferometer with equal junctions of negligible capacitance, as shown in Figure 1. The dynamical equations for the variables φ and ψ, characterizing this system, can be written in the following form [6]:

$$\frac{d\varphi}{d\tau} + (-1)^n \cos(\pi\psi) \sin\varphi = \frac{i_B}{2}; \tag{3a}$$

$$\pi\frac{d\psi}{d\tau} + (-1)^n \sin(\pi\psi) \cos\varphi + \frac{\psi}{2\beta} = \frac{\psi_{ex}}{2\beta}. \tag{3b}$$

Let us consider a new time variable $\vartheta = \frac{\tau}{2\pi\beta} = \frac{R}{L}t$ and write the solutions for φ and ψ in the following form: $\varphi(\beta,\tau) \approx \varphi_{0,\beta}(\tau) + \beta\varphi_{1,\beta}(\tau)$; $\psi(\beta,\tau) \approx \psi_0(\vartheta) + \beta\psi_1(\vartheta)$. This approach allows us not only to account for the regular part of the solution, as seen in references [6,7], but also to consider its singular part. Moreover, as we shall see, the role of the two time variables will become evident in what follows, since one time scale is defined for equation (3a) and one for equation (3b). Consider then $\varphi(\beta,\tau)$ and $\psi(\beta,\tau)$ to be bounded, differentiable functions, and expand the sine and cosine functions appearing in equations (3a, 3b) to first order in β. By then substituting equations (5) and (6) in equations (3a) and (3b), respectively, and by collecting all coefficients of identical power of β, we can obtain a system of equations for the functions $\varphi_{k,\beta}(\tau)$ and $\psi_k(\vartheta)$, with $k = 0,\ 1$. These functions are determined according to the following sequential scheme. As a first step, we use equation (3b) to determine $\psi_0(\vartheta)$. We adopt the solution found and substitute it in equation (3a) to determine $\varphi_{0,\beta}(\tau)$. The latter solution, on its turn, is substituted in equation (3b) to find $\psi_1(\vartheta)$ and, finally, this solution is used in equation (3a) to find $\varphi_{1,\beta}(\tau)$. Note, however, that for defining first order solutions, knowledge of zero-th order solutions is required. Furthermore, we assume that the initial conditions are the following:

$$\varphi(\beta,0) = \varphi_{0,\beta}(0) + \beta\varphi_{1,\beta}(0); \quad \psi(\beta,0) = \psi_0(0) + \beta\psi_1(0). \tag{4}$$

As for initial conditions, from equation (3b) we may notice that $\psi_0(\tau) = \psi_{ex}$ for $\beta = 0$, in which case we cannot even define the time variable ϑ. This condition, however, is inherited by the function $\psi_0(\vartheta)$, since the following equalities are satisfied:

$$\psi_0\left(\frac{\tau}{\beta}\right)_{\beta=0} = \psi_0(\vartheta)_{\vartheta\to\infty} = \psi_{ex}. \tag{5}$$

Furthermore, we may also notice that $\psi_k(\vartheta)_{\vartheta=0} = \psi_k\left(\frac{\tau}{\beta}\right)_{\tau=0} = \psi_k(0)$, for $k = 0, 1$.

By the general procedure described above we get the following differential equations for the superconducting phase variables:

$$\frac{d\varphi_{0,\beta}}{d\tau} + (-1)^n \cos(\pi\psi_0(\vartheta)) \sin\varphi_{0,\beta}(\tau) = \frac{i_B}{2}, \tag{6a}$$

$$\frac{d\varphi_{1,\beta}}{d\tau} + (-1)^n \varphi_{1,\beta}(\tau) \cos(\pi\psi_0(\vartheta)) \cos\varphi_{0,\beta}(\tau) = (-1)^n \pi\psi_1(\vartheta) \sin(\pi\psi_0(\vartheta)) \sin\varphi_{0,\beta}(\tau); \tag{6b}$$

and the following for the flux number variables:

$$\frac{d\psi_0}{d\vartheta} + \psi_0(\vartheta) = \psi_{ex}, \tag{7a}$$

$$\frac{d\psi_1}{d\vartheta} + \psi_1(\vartheta) = 2(-1)^{n-1} \sin(\pi\psi_0(\vartheta)) \cos\varphi_{0,\beta}(2\pi\beta\vartheta). \tag{7b}$$

In equations (6a, 6b) and (7a, 7b) we may notice the appearance of two different time scales the first, $\tau_\psi = \frac{L}{R}$, linked to flux motion in and out the superconducting ring, the second, $\tau_\varphi = \frac{\Phi_0}{2\pi R I_J}$, pertaining to the dynamics of the superconducting phase difference value φ. We notice that $\frac{\tau_\psi}{\tau_\varphi} = 2\pi\beta$, so that, for negligible values of this ratio, the system behaves effectively as if an adiabatic time evolution of the superconducting phase difference variable φ could be studied under the assumption that asymptotic solutions of ψ could be used. In this case, therefore, we may first let the system evolve in its flux states, so that a stationary magnetic state is reached, and then solve for the superconducting phase difference time evolution of the system. This is exactly what is done, under the assumptions of negligible value of the ratio $\frac{\tau_\psi}{\tau_\varphi}$, in references [6,7]. However, when one would like to acquire the regular solution for the system dynamics, even when considering the approximate solution for the variable φ and ψ, one needs to follow the more general perturbation analysis described in the present work, where the ratio might not a priori considered as negligible. Finally, considering that this ratio is proportional to the perturbation parameter, one might wish to generalize the above analysis to higher order in the parameter β, in order to have a wider range of validity of the analysis itself. Therefore, this analysis could be extended to higher order approximations of the perturbation solutions, and this will be done in a future work.

As remarked before, by limiting our analysis to the case in which the parameter β is considered small, in what follows we shall only be concerned with a single time scale, namely τ_φ, by assuming that the transient of the flux variable rapidly vanishes ($\frac{\tau_\psi}{\tau_\varphi} \ll 1$). In this way, the dynamical equations for the flux variable (Eqs. (7a, 7b)) give the following steady-state solution for ψ_0 and ψ_1:

$$\psi_0 = \psi_{ex}, \tag{8a}$$

$$\psi_1 = 2\,(-1)^{n-1} \sin(\pi\psi_{ex}) \cos\varphi_0 - 2\pi \frac{\mathrm{d}\psi_{ex}}{\mathrm{d}\tau}, \tag{8b}$$

where the term $2\pi\frac{\mathrm{d}\psi_{ex}}{\mathrm{d}\tau}$ has been inserted in equation (8b), in order to correctly take account of first order contributions in β, when considering only the time scale τ_φ and the subscript β in $\varphi_{0,\beta}(\tau)$ has been elided, as it will be done for $\varphi_{1,\beta}(\tau)$ from this point on, since these functions will not depend on β in this limit.

3 Constant applied flux

In the present section we consider ψ_{ex} as constant. By the general procedure schematized above we get the following differential equations for the superconducting phase variables:

$$\frac{d\varphi_0}{d\tau} + (-1)^n \cos(\pi\psi_{ex}) \sin\varphi_0(\tau) = \frac{i_B}{2}, \tag{9a}$$

$$\frac{d\varphi_1}{d\tau} + (-1)^n \varphi_1(\tau) \cos(\pi\psi_{ex}) \cos\varphi_0(\tau) = (-1)^n \pi\psi_1(\tau) \sin(\pi\psi_{ex}) \sin\varphi_0(\tau)\,; \tag{9b}$$

and the following for the flux number variables:

$$\psi_0 = \psi_{ex}, \tag{10a}$$

$$\psi_1(\tau) = 2\,(-1)^{n-1} \sin(\pi\psi_{ex}) \cos\varphi_0(\tau)\,. \tag{10b}$$

According to the scheme described in the previous section, by having already set $\psi_0 = \psi_{ex}$ in equation (9a) we may solve for $\varphi_0(\tau)$.

We now briefly discuss how to obtain this solution. In the case

$$\left|\frac{i_B}{2(-1)^n \cos(\pi\psi_{ex})}\right| = \frac{1}{|a|} > 1,$$

which characterizes the running state of the junctions, we have

$$\varphi_0(\tau) = 2\tan^{-1}\left[a + \sqrt{1-a^2}\tan\left(\gamma\tau + \tan^{-1}\xi_0\right)\right], \tag{11}$$

where

$$\gamma = \frac{1}{2}\sqrt{\frac{i_B^2}{4} - \cos^2(\pi\psi_{ex})}$$

and

$$\xi_0 = \frac{\tan\left(\frac{\varphi_0(0)}{2}\right) - a}{\sqrt{1-a^2}}.$$

On the other hand, in the case

$$\left|\frac{i_B}{2(-1)^n \cos(\pi\psi_{ex})}\right| = \frac{1}{|a|} < 1,$$

which characterizes the superconducting state of the junctions, we have

$$\varphi_0(\tau) = 2\tan^{-1}\left[a + \sqrt{a^2-1}\tanh\left(\tilde{\gamma}\tau + \tanh^{-1}(\chi_0)\right)\right], \tag{12}$$

where

$$\tilde{\gamma} = \frac{1}{2}\sqrt{\cos^2(\pi\psi_{ex}) - \frac{i_B^2}{4}}$$

and

$$\chi_0 = \frac{\tan\left(\frac{\varphi_0(0)}{2}\right) - a}{\sqrt{a^2-1}}.$$

Finally, in the case $|a| = 1$, we have

$$\varphi_0(\tau) = \mathrm{sgn}(a)\left[2\tan^{-1}\left(\frac{i_B\tau}{2} + \omega_0^{(\pm)}\right) - \frac{\pi}{2}\right], \tag{13}$$

where $\omega_0^{(\pm)} = \tan\left(\mathrm{sgn}(a)\,\frac{\varphi_0(0)}{2} + \frac{\pi}{4}\right)$.

Having found the time dependence of the variable φ_0, ψ_1 can be found by equation (10b) by substitution. Finally, by knowledge of ψ_0, φ_0 and ψ_1, φ_1 can be found by equation (9b), which is a standard first order linear differential equation. Solutions for φ_0 are shown in Figures 2a–2c for $\cos(\pi\psi_{ex}) = 0.3$ and $i_B = 1.6$, $i_B = 0.4$ and $i_B = 0.6$, respectively, along with the solution obtained by numerically integrating equations (3a, 3b). In Figures 2b–2c the first order approximation of the solution is shown as a dotted line. Solutions for φ_1 are shown in Figures 3a–3c for $\cos(\pi\psi_{ex}) = 0.3$ and $i_B = 1.6$, $i_B = 0.4$ and $i_B = 0.6$, respectively. The above analysis thus leads to a solution in a closed form, to first order in the parameter β. Notice that in the case of time-dependent bias currents one should adopt a more general procedure.

As a simple application, let us calculate, to first order in the parameter β, the circulating current i_S in the circuit, normalized to I_J, given by [1]:

$$i_S = \frac{\psi - \psi_{ex}}{\beta}. \tag{14}$$

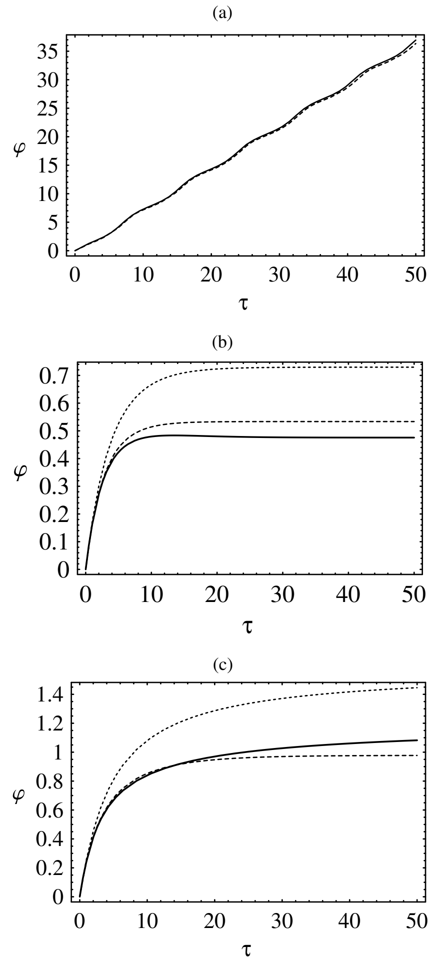

Fig. 2. Average phase difference by setting $\cos(\pi\psi_{ex}) = 0.3$ and: (a) $i_B = 1.6$; (b) $i_B = 0.4$; (c) $i_B = 0.6$. Dotted lines represent $\varphi_0(\tau)$ as calculated to zero-th order in the parameter β (taken to be 0.02), full lines represent $\varphi(\tau)$ as calculated to first order in the parameter β (taken to be 0.02), and dashed lines represents the numerical solution of the complete system. In (a) the first order approximation of the solution is not shown, for clarity reasons.

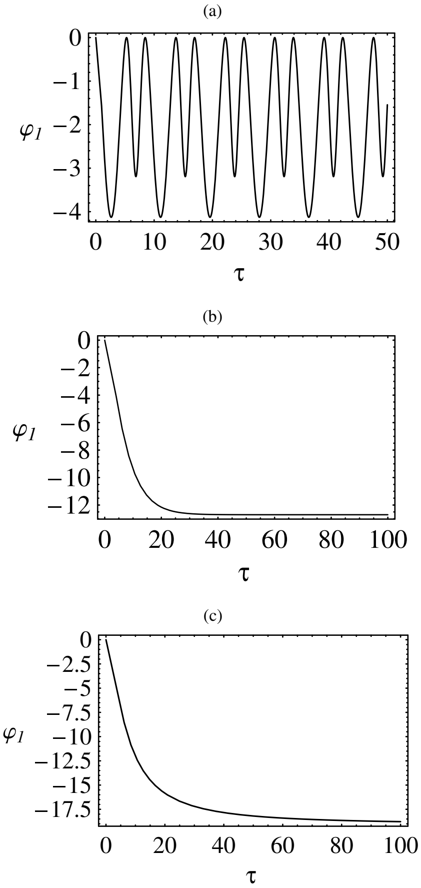

Fig. 3. First order correction to $\varphi_0(\tau)$ as calculated by the procedure described in the text by setting $\beta = 0.02$ $\cos(\pi\psi_{ex}) = 0.3$ and: (a) $i_B = 1.6$; (b) $i_B = 0.4$; (c) $i_B = 0.6$.

for $\psi_{ex} = 0.3$ and 0.7, although having the same period, as it can be argued from equation (16), are not equal. Finally notice also that, by equation (15), for odd values of n the current just reverses its sign.

For an arbitrary value n, which represents the number of fluxons initially trapped in the superconducting ring, we have

$$i_S = \psi_1(\tau) = 2(-1)^{n-1}\sin(\pi\psi_{ex})\cos\varphi_0(\tau). \quad (15)$$

Graphs of circulating currents are shown in Figures 4a, 4b, 4c for n even, $i_B = 2.2$ and for $\psi_{ex} = 0.1$ and 0.3, $\psi_{ex} = 0.5$, $\psi_{ex} = 0.7$ and 0.9, respectively. The period T of these curves is equal to the pseudo-period of $\varphi_{0,\beta}$ which is given by the following expression in terms of ψ_{ex} and i_B:

$$T = \frac{2\pi}{\sqrt{\left(\frac{i_B}{2}\right)^2 - \cos^2(\pi\psi_{ex})}}. \quad (16)$$

Notice that the lowest value of the period is obtained for $\psi_{ex} = 0.5$ and that the curves for $\psi_{ex} = 0.1$ and 0.9 and

4 Time-dependent applied flux

The results in the previous section have been obtained for the magnetic response of the system in the presence of a constant applied flux. In the present section we shall analyze the electrodynamic response of the two junction quantum interferometer in the presence of a time-dependent external flux. For this purpose, we shall take a sinusoidal forcing term, in such a way that $\psi_{ex}(t) = A + B\cos(\omega t)$, where A is the normalized d.c. component of the applied flux and B is the normalized amplitude of the a.c. signal. Now, since $t = \frac{\Phi_0}{2\pi RI_J}\tau$, we can write

$$\psi_{ex}(\tau) = A + B\cos(\tilde{\omega}\tau), \quad (17)$$

where $\tilde{\omega} = \frac{\Phi_0}{2\pi RI_J}\omega$. We shall assume that the normalized amplitude B of the oscillating signal is much less than one

$(B \ll 1)$. The perturbation analysis is then carried out in a similar way as in the previous section.

We start by setting, by equations (8a) and (8b) $\psi_0(\tau) = \psi_{ex}(\tau) = A + B\cos(\tilde{\omega}\tau)$ and $\psi_1 = 2(-1)^{n-1}\sin(\pi\psi_{ex}(\tau))\cos\varphi_0 - 2\pi\frac{d\psi_{ex}}{d\tau}$ and solve the equations for the phase differences

$$\frac{d\varphi_0}{d\tau} + (-1)^n\cos(\pi\psi_{ex}(\tau))\sin\varphi_0(\tau) = \frac{i_B}{2}, \quad (18a)$$

$$\frac{d\varphi_1}{d\tau} + (-1)^n\varphi_1(\tau)\cos(\pi\psi_{ex}(\tau))\cos\varphi_0(\tau) = -\pi\sin^2(\pi\psi_{ex}(\tau))\sin(2\varphi_0(\tau)). \quad (18b)$$

By noticing, however, that $\cos(\pi\psi_{ex}(\tau)) = \cos(\pi A + \pi B\cos(\tilde{\omega}\tau)) \approx \cos(\pi A) - \pi B\sin(\pi A)\cos(\tilde{\omega}\tau)$, equation (18a) can be written in the following form:

$$\frac{d\varphi_0}{d\tau} + (-1)^n(a - b\cos(\tilde{\omega}\tau))\sin\varphi_0(\tau) = \frac{i_B}{2}, \quad (19)$$

where $a = \cos(\pi A)$ and $b = \pi B\sin(\pi A)$. In equation (19) we find a perturbed solution in terms of the parameter b, so that, by setting $\varphi_0(\tau) = \eta_0(\tau) + b\eta_1(\tau)$, we can write:

$$\frac{d\eta_0}{d\tau} + (-1)^n a\sin\eta_0(\tau) = \frac{i_B}{2}, \quad (20a)$$

$$\frac{d\eta_1}{d\tau} + (-1)^n a\cos\eta_0(\tau)\eta_1(\tau) = (-1)^n\sin\eta_0(\tau)\cos(\tilde{\omega}\tau). \quad (20b)$$

Notice then that the solutions to the above equations can be found by exactly the same procedure described in the previous section. Once the solution for $\varphi_0(\tau)$ is found, by substituting in equation (18b), the solution for $\varphi_1(\tau)$ can be determined by solving a first order linear differential equation with time-dependent coefficients. Assuming thus $\varphi_0(\tau) = \eta_0(\tau) + b\eta_1(\tau)$ to be a known expression, we can then write:

$$i_S = 2(-1)^{n-1}\sin(\pi\psi_{ex}(\tau))\cos(\eta_0(\tau) + b\eta_1(\tau)) + 2\pi\tilde{\omega}B\sin\tilde{\omega}\tau. \quad (21)$$

As in the previous section, the above expression, equal to $\psi_1(\tau)$, represents the circulating current i_S in the circuit. In Figures 5a–5c we represent the time dependence of the current i_S for the respective values of the normalized frequency $\tilde{\omega} = 0.03$, 0.06, 0.09, and for $A = 0$, $\beta = 0.01$ and $i_B = 2.5$. In these graphs we notice that the oscillating patterns, which we have already detected in the constant applied field case, are modulated by the externally applied oscillating signal.

Another important quantity to be measured in these systems is the critical current i_c, which is the maximum value of the current bias i_B which can be injected in the two junction interferometer without giving rise to dissipation. By considering the stationary case of equation (21a) we write:

$$i_B = 2(-1)^n\cos(\pi\psi_{ex}(\tau))\sin\varphi_0(\tau). \quad (22)$$

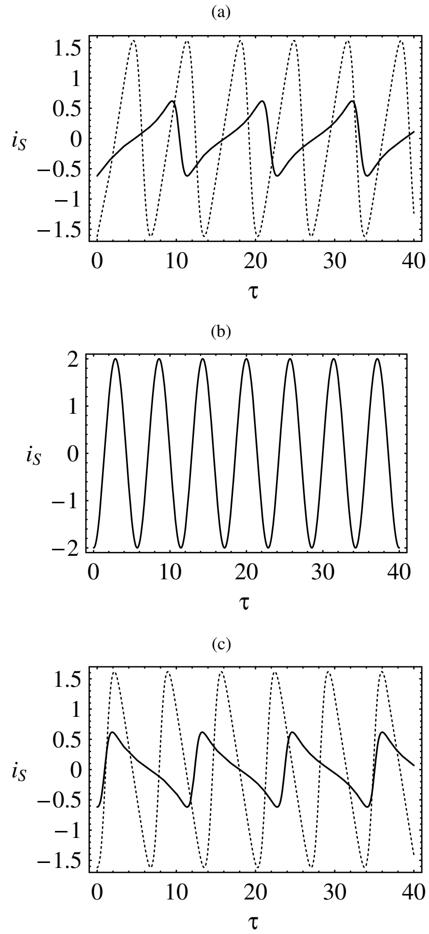

Fig. 4. Circulating current i_S as a function of the normalized time for null values of the initially trapped flux and for $i_B = 2.2$ and: (a) $\psi_{ex} = 0.1$ (full line), $\psi_{ex} = 0.3$ (dashed line); (b) $\psi_{ex} = 0.5$; (c) $\psi_{ex} = 0.7$ (dashed line), $\psi_{ex} = 0.9$ (full line).

Therefore, we have

$$i_c = 2\left|\cos(\pi A + \pi B\cos(\tilde{\omega}\tau))\right|. \quad (23)$$

Noticing that the time-averaged value $\langle i_c \rangle$ of the critical current do not depend on the normalized frequency, it can be calculated in terms of solely A and B, the results being shown in Figures 6a and 6b for null values of the initially trapped flux. In particular, in Figure 6a $\langle i_c \rangle$ is shown as a function of the applied magnetic field amplitude B, for $A = 0.1$ and $A = 0.4$, while in Figure 6b, $\langle i_c \rangle$ vs. A curves are shown for $B = 0.1$ and $B = 0.2$. In the curves in Figure 6a we notice Fraunhofer-like oscillations, while ordinary cosinusoidal oscillations are present in Figure 6b.

5 Conclusions

We studied the dynamical properties of a symmetric quantum interferometer containing two identical junctions with negligible capacitance by means of a perturbation approach in the parameter β, whose value gives the strength

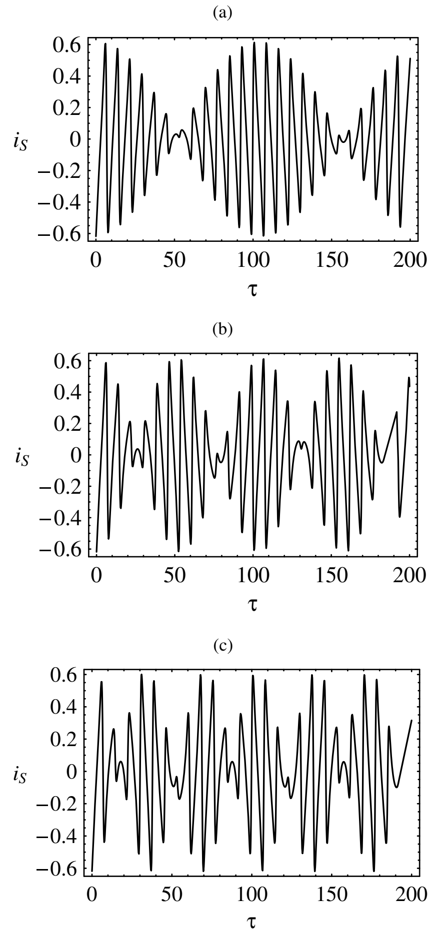

Fig. 5. Normalized time dependence of the circulating current i_S in the presence of an oscillating field with $A = 0$, $B = 0.1$ and normalized frequencies equal to: (a) $\tilde{\omega} = 0.03$; (b) $\tilde{\omega} = 0.06$; (c) $\tilde{\omega} = 0.09$. No initially trapped flux in the SQUID is assumed and $\beta = 0.01$, $i_B = 2.5$.

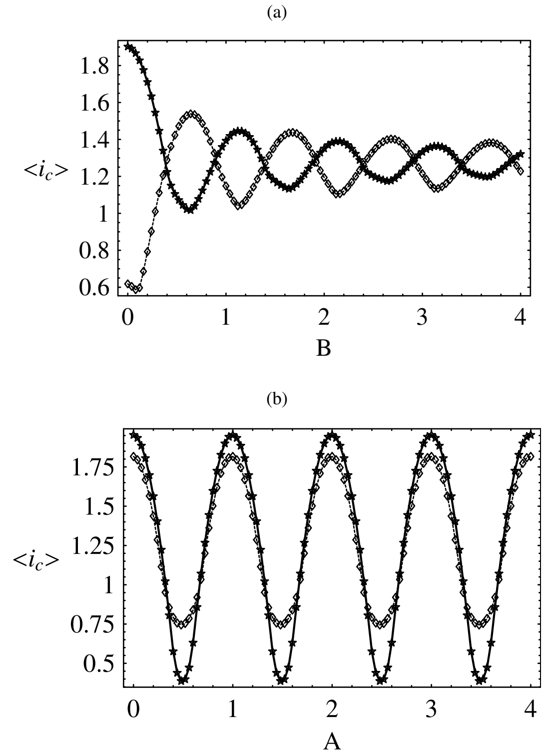

Fig. 6. (a) Time average value $\langle i_c \rangle$ of the critical current i_c as a function of the amplitude B of the oscillating magnetic flux for $A = 0.1$ and $A = 0.2$, for null values of the initially trapped flux. The curve with stars is plotted for $A = 0.1$, the curve with diamond is plotted for $A = 0.2$. (b) Time average value $\langle i_c \rangle$ of the critical current i_c as a function of A for null values of the initially trapped flux, and for two values of the amplitude B of the oscillating magnetic flux. The curve with stars is plotted for $B = 0.1$, the curve with diamond is plotted for $B = 0.4$.

of the electromagnetic coupling between the two junction in the system. The analysis is rather similar to what done in other works in the literature [6,7]. However, in the present work we present a rather general procedure to obtain the solution to the problem to first order in the parameter β. Considering at first transient solutions, we have noticed that the function $\psi(\beta, \vartheta)$ governs fluxon dynamics, where ϑ is the ordinary time t, normalized to the characteristic circuital time constant $\tau_\Psi = \frac{L}{R}$. By this more general approach it becomes thus evident that the characteristic time constant $\tau_\varphi = \frac{\Phi_0}{2\pi R I_J}$ of the dynamics of the average superconducting phase difference φ is different from the fluxon dynamics characteristic time τ_Ψ, so that the asymptotic solution for the system, proposed in the analyses carried out in references [6,7], takes on a more precise meaning in this context. Indeed, when the parameter β is sufficiently small to allow, for finite values of τ, an asymptotic evaluation of $\psi_0\left(\frac{\tau}{2\pi\beta}\right)$ and $\psi_1\left(\frac{\tau}{2\pi\beta}\right)$, the general solution given in the present work coincides with the asymptotic perturbed solution proposed in references [6,7] in the limit of negligible junction capacitance.

The perturbation analysis has been first carried out for a constant applied magnetic flux. Successively, since it could be experimentally possible to force the system with a time-dependent magnetic field, it is noted that the perturbed solution for the flux number ψ, obtained for a sinusoidal magnetic flux, needs careful evaluation. In order to exhibit experimentally detectable quantities, the circulating current i_S is evaluated as a function of time, for different values of the frequency of the forcing field. Finally, the time average $\langle i_c \rangle$ of the critical current of the device has been studied both as a function of the d.c. component and of the amplitude of the oscillating part of the applied flux. In these curves two characteristic behaviours have been detected: a Fraunhofer-like pattern in $\langle i_c \rangle$ vs. B curves; independence of $\langle i_c \rangle$ from $\tilde{\omega}$.

All these results could trigger experimental studies on this well known and widely used device, especially considering that these systems can be adopted as elementary models for granular superconductors [10], where the coupling between adjacent superconducting grains is described by means of Josephson links.

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